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ONE-DIMENSIONAL FREEZING OR MELTING PROCESS IN A BODY WITH VARIABLE CROSS-SECTIONAL AREA

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NOMENCLATURE

- a, thermal diffusivity;
- A, cross-sectional area;
- L, latent heat of fusion;
- q, heat flux;
- t, time;
- T, temperature;
- T_c, temperature of a cold medium;
- T_0, temperature given at the position x_0 for the boundary condition of the first kind;
- T_s, solidification temperature;
- x, position coordinate;
- x_0, position, through which heat is removed;
- y, transformed position coordinate defined in equation (6);
- α, heat-transfer coefficient;
- η, transformed coordinate of the phase boundary defined in equation (7);
- λ, thermal conductivity;
- ξ, coordinate of the phase boundary;
- ρ, density in the solid phase.

1. INTRODUCTION

THE APPLICATION of solutions to the unsteady heat conduction problem with change of phase is of great practical importance. For example, the problem of ice formation is very significant both in geophysics and in ice manufacture. A great deal of attention has been given to the solidification of castings. In aerospace studies the ablation problem is encountered when a body moves with hypersonic velocity through the earth's atmosphere. The characteristic feature of such problems is the coupling of the temperature field with the rate of propagation of the phase boundary between the solid and liquid phases, which makes the problem non-linear. Only a few exact analytical solutions have been found for special cases. For example, the Neumann's solution (see Carslaw and Jaeger [1]) relates to the solidification of the semi-infinite region of liquid, initially above or at the fusion temperature, when the surface wall temperature is suddenly decreased to and maintained at a temperature below fusion.

With regard to the non-linear problem in which the position of the phase boundary is unknown, solutions are expected to be obtained by analytical approximations and numerical methods. For example, the heat balance integral method was used by Goodman [2], Goodman and Shea [3] and Poots [4]; the variational method by Biot [5] and Biot and Daughaday [6]; the method of moving heat sources by Rosenthal [7] and Jackson [8]. Lin [9] used a transformation to obtain relations for calculation of the rate of propagation of the phase boundary in cylindrical and spherical coordinates from that in Cartesian coordinates. Using the polynomial approximation, Megerlin [10] derived equations for this rate in the planar case with different boundary conditions, and then used the relations in [9] to obtain the cylindrical and spherical cases. An excellent review in the field of heat conduction with freezing or melting was given by Muehlbauer and Sunderland [11].

In the present work, the relations in [9] are extended. We consider a freezing or melting process, which takes place in a body with a variable cross-sectional area. We denote the length of the central line of the body by x, and assume that (1) the body is insulated from the surroundings, (2) the radius of curvature of the central line is very large and (3) the change of cross-sectional area with x is relatively small. With these assumptions, the temperature distribution across the central line can be taken as constant. Then the problem can be treated as one-dimensional. In order to deal with the problem easily, we assume further that the densities in the liquid and solid phases are kept constant and that convection in the liquid phase is negligible.

The freezing and the melting process can be described by the same mathematical formulations. In the following, only the freezing process will be discussed. The results are applicable for the melting process.

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2. ONE-DIMENSIONAL FREEZING PROCESS

To simplify the problem, the entire fluid is assumed at the beginning of the freezing process to be at the solidification temperature, T_s. All of the material properties are taken to be constant.

The equation of one-dimensional heat conduction can be easily derived as [12]

$$\frac{\partial T}{\partial t} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{A'(x)}{A(x)}\frac{\partial T}{\partial x}\right), \tag{1}$$

where T is the temperature, t the time, a the thermal diffusivity, x the position coordinate, $A(x)$ the cross-sectional area, and $A'(x)$ the first derivative of $A(x)$. The initial condition is obvious, for $x \geqslant x_0$ and $t = 0$:

$$T(x, 0) = T_s. \tag{2}$$

The boundary conditions at the phase boundary $x = \xi(t)$ for $t > 0$ are

$$\frac{d\xi}{dt} = \frac{\lambda}{\rho L}\frac{\partial T(\xi, t)}{\partial x} \tag{3}$$

and

$$T(\xi, t) = T_s, \tag{4}$$

where λ and ρ are the thermal conductivity and density in the solid phase, respectively, and L is the latent heat of fusion.

For the boundary condition at the surface $x = x_0$, through which heat is removed, any one of the following conditions may apply: for

$$x = x_0 \quad \text{and} \quad t > 0:$$

$$T(x_0, t) = T_0, \tag{5.1}$$

boundary condition of the first kind given by the surface temperature T_0,

$$\lambda\frac{\partial T(x_0, t)}{\partial x} = q, \tag{5.2}$$

boundary condition of the second kind given by the heat flux q, or

$$\lambda\frac{\partial T(x_0, t)}{\partial x} = \alpha[T(x_0, t) - T_c], \tag{5.3}$$

boundary condition of the third kind given by the heat transfer with convection, where α is the heat transfer coefficient and T_c is the temperature of the cold medium.

To solve this problem, we must find the solution of the differential equation (1), which satisfies the initial and boundary conditions (2)–(5). However, the boundary condition (3) is an unknown function which depends upon the temperature gradient at the phase boundary, or in other words, depends upon the solution of the problem. This therefore makes the problem very complicated.

In practice, however, we are especially interested in the rate of propagation of the phase boundary, or simply, the interface velocity $d\xi/dt$ [equation (3)]. Hereafter, we will concentrate on solving the problem of this interface velocity.

The method of solving the problem of the interface velocity consists of three steps: (1) transformation of the position coordinate, (2) description of the temperature field and (3) consideration of the temperature distribution in the neighbourhood of the phase boundary.

2.1 *Transformation of the position coordinate*

For the transformation of the position coordinate, the following equation is used:

$$y = A(x_0)\int_{x_0}^{x}\frac{ds}{A(s)}. \tag{6}$$

The coordinate of the phase boundary ξ is similarly transformed by

$$\eta = A(x_0)\int_{x_0}^{\xi}\frac{ds}{A(s)}. \tag{7}$$

With these transformations, we obtain from equations (1)–(5) a new system of equations, which describe the problem:

$$\frac{\partial T}{\partial t} = a\left[\frac{A(x_0)}{A(x)}\right]^2\frac{\partial^2 T}{\partial y^2}, \tag{8}$$

$$T(y, 0) = T_s, \tag{9}$$

$$\frac{d\eta}{dt} = \frac{\lambda}{\rho L}\left[\frac{A(x_0)}{A(\xi)}\right]^2\frac{\partial T(\eta, t)}{\partial y}, \tag{10}$$

$$T(\eta, t) = T_s, \tag{11}$$

and

$$T(0, t) = T_0, \tag{12.1}$$

$$\lambda\frac{\partial T(0, t)}{\partial y} = q \tag{12.2}$$

or

$$\lambda\frac{\partial T(0, t)}{\partial y} = \alpha[T(0, t) - T_c]. \tag{12.3}$$

2.2 *Description of the temperature field*

The temperature T is a function of position y and time t. Between the time t and the coordinate of the phase boundary η, there is a certain relation with $\eta = \eta(t)$ or inversely $t = t(\eta)$. Therefore the temperature function can be written

$$T = f[y, t(\eta)],$$

or

$$T = T[y, \eta(t)]. \tag{13}$$

The selection of the coordinates y and $\eta(t)$ for the temperature function has two advantages; one is that the position of the phase boundary in the $y - \eta$ plane is simply a straight line with $y = \eta$ (Fig. 1), and the other is that the derivative of the temperature function with respect to time is equal to

$$\frac{\partial T}{\partial t} = \frac{\partial T}{\partial \eta}\frac{\mathrm{d}\eta}{\mathrm{d}t}. \tag{14}$$

This means that the derivative of the temperature function with respect to time contains the interface velocity, which has the form in the $y - \eta$ plane,

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = \frac{\lambda}{\rho L}\left[\frac{A(x_0)}{A(\xi)}\right]^2 \frac{\partial T(\eta,\eta)}{\partial y}. \tag{15}$$

If we put the equation of interface velocity (15) into equation (14), and then put equation (14) into the differential equation (8), we have

$$\frac{\partial^2 T}{\partial y^2} = \frac{\lambda}{a\rho L}\left[\frac{A(x)}{A(\xi)}\right]^2 \frac{\partial T(\eta,\eta)}{\partial y}\frac{\partial T}{\partial \eta}. \tag{16}$$

This is clearly non-linear.

2.3 *Consideration of the temperature distribution in the neighbourhood of the phase boundary*

From equation (15) we see that the interface velocity depends only upon the temperature gradient at the phase boundary. If the temperature distribution in the neighbourhood of the phase boundary is known, then the interface velocity can be determined. We consider the temperature distribution in the neighbourhood of the phase boundary in the regions $\xi - \Delta x \leqslant x \leqslant \xi$ and $\eta - \Delta y \leqslant y \leqslant \eta$ for the coordinate systems before and after the transformation of position coordinate, respectively (Fig. 1). The function of the cross-sectional area in the neighbourhood of the phase boundary can be obtained with the Taylor's series,

$$A(x) = A(\xi - \varepsilon) = A(\xi) - \varepsilon A'(\xi) + \ldots \quad \text{with} \quad 0 \leqslant \varepsilon \leqslant \Delta x. \tag{17}$$

We have the factor $[A(x)/A(\xi)]^2$ in the neighbourhood of the phase boundary, and with the limit $\Delta x \to 0$:

$$\lim_{\Delta x \to 0}\left[\frac{A(x)}{A(\xi)}\right]^2 = \lim_{\varepsilon \to 0}\left[1 - \varepsilon\frac{A'(\xi)}{A(\xi)} + \ldots\right]^2 = 1. \tag{18}$$

We then have the system of equations (16), (9), (11) and (12) in the neighbourhood of the phase boundary in the $y - \eta$ plane:

$$\frac{\partial^2 T}{\partial y^2} = \frac{\lambda}{a\rho L}\frac{\partial T(\eta,\eta)}{\partial y}\frac{\partial T}{\partial \eta}, \tag{19}$$

$$T(y,0) = T_s, \tag{20}$$

$$T(\eta,\eta) = T_s, \tag{21}$$

and

$$T(0,\eta) = T_0, \tag{22.1}$$

$$\lambda\frac{\partial T(0,\eta)}{\partial y} = q \tag{22.2}$$

$$\lambda\frac{\partial T(0,\eta)}{\partial y} = \alpha[T(0,\eta) - T_c]. \tag{22.3}$$

This system of equations is now independent of the cross-sectional area. The solution of the temperature T and of the temperature gradient $\partial T/\partial y$ of this system must be *universal functions* which are available for all kinds of cross-sectional areas.

The interface velocity is described by the equation (15). For the special case $A(x) = \text{const.}$, we have

$$\left(\frac{\mathrm{d}\eta}{\mathrm{d}t}\right)_{A(x)=\text{const.}} = \frac{\lambda}{\rho L}\frac{\partial T(\eta,\eta)}{\partial y} = g(\eta). \tag{23}$$

We then have the interface velocity for a variable cross-

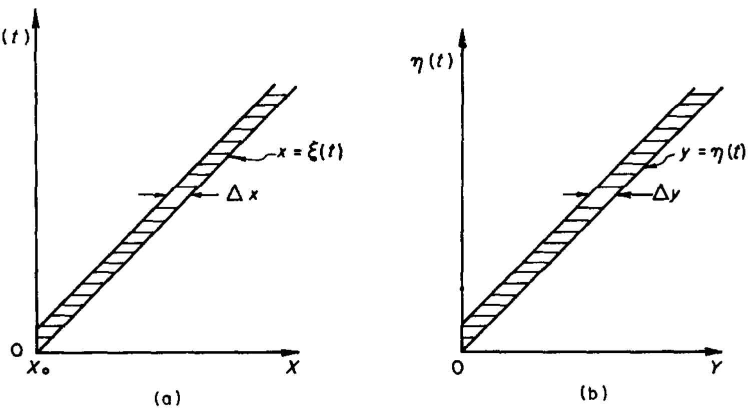

FIG. 1. Regions for the temperature distribution in the neighbourhood of the phase boundary. (a) and (b) are the coordinate systems before and after the transformation of position coordinate, respectively.

sectional area $A(x)$,

$$\left(\frac{d\eta}{dt}\right)_{A(x)} = \left[\frac{A(x_0)}{A(\xi)}\right]^2 g(\eta). \tag{24}$$

3. CONCLUSIONS

From the inverse transformation, equation (7), we obtain a theorem for calculation of the interface velocity of a freezing or melting process in a body with variable cross-sectional area:

Theorem: If the interface velocity of a freezing or melting process in a body with constant cross-sectional area and with $x_0 = 0$ is

$$\left(\frac{d\xi}{dt}\right)_{A(x)=\text{const.}} = g(\delta) \tag{25}$$

then the interface velocity of the freezing or melting process in a body with variable cross-sectional area $A(x)$ is

$$\left(\frac{d\xi}{dt}\right)_{A(x)} = \frac{A(x_0)}{A(\xi)} g\left(A(x_0)\int_{x_0}^{\xi}\frac{ds}{A(s)}\right), \tag{26}$$

provided that the initial and boundary conditions, with the exception of the interface velocity, after the transformation of position coordinate, equation (6), for both these cases are the same.

It is obvious that, using the same process as above, we can prove that this theorem is also true for other initial and boundary conditions; for example, where the initial temperature is different from the solidification temperature and another boundary condition takes place at a surface elsewhere, $x = x_1$.

For special cases in cylindrical and spherical coordinate systems with $A(x)_{\text{cyl.}} = 2\pi l x$ and $A(x)_{\text{sphere}} = 4\pi x^2$, we obtain

$$\left(\frac{d\xi}{dt}\right)_{\text{cyl.}} = \frac{x_0}{\xi} g\left(x_0 \ln\frac{\xi}{x_0}\right) \tag{27}$$

and

$$\left(\frac{d\xi}{dt}\right)_{\text{sphere}} = \left(\frac{x_0}{\xi}\right)^2 g\left(x_0 - \frac{x_0^2}{\xi}\right). \tag{28}$$

These results are the same as in [9].

The exact solution of the interface velocity in a body with constant cross-sectional area for the boundary condition of the first kind, equation (5.1), is well known as Neumann's solution [1]. There is no exact solution for the boundary conditions of the second and third kinds, equations (5.2) and (5.3). However, the approximations of Goodman [2] and Magerlin [10], for calculation of the interface velocity for $A(x)$ = const. with these boundary conditions, are available. As the interface velocity for constant cross-sectional area is known, the interface velocity for variable cross-sectional area can therefore be calculated.

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